Designing a Zip-line

Problem:

We want to attach the cable to a high and low point so that someone can enjoy the benefits of gravity and slide down, but we need just enough sag in the cable to enable them to reach the landing at a reasonable speed.

What should the relative position of the start/finish be and how long should the cable be?

The ride we must take is a lot more frightening than riding a Zipline ... let’s go!
Given: $W =$ width between the supports  
$H =$ height between the supports  
$s =$ arc length of the cable  

Find: an equation for the catenary curve  

Solution:

(1) $y = \cosh \left( \frac{x}{a} \right) - a$ ... the equation of the catenary passing through the origin ... so all we need is a value for $a$ ... hang on, here we go!

(2) $y_1 = \cosh \left( \frac{x_1}{a} \right) - a$ ... one solution

(3) $y_2 = \cosh \left( \frac{x_2}{a} \right) - a$ ... another solution

(4) $y_1 + H = y_2$

(5) $x_2 + W = x_1$

(6) $\cosh^2 t - \sinh^2 t = 1$ ... hyperbolic identity

(7) $s = \int_{x_2}^{x_1} \sqrt{1 + \sinh^2 \left( \frac{x}{a} \right)} \quad \text{... the arc length formula from calculus}$

(8) $s = \sinh \left( \frac{x_1}{a} \right) - \sinh \left( \frac{x_2}{a} \right)$ ... evaluating integral (7) using identity (6)

(9) $s = \sinh \left( \frac{x_2 + W}{a} \right) - \sinh \left( \frac{x_2}{a} \right)$ ... combining (5) & (8)

(10) $x_1 = \sinh^{-1} \left( \frac{s}{a} + \sinh \left( \frac{x_2}{a} \right) \right)$ ... solving (8) for $x_1$

(11) $y_1 = \cosh \left( \sinh^{-1} \left( \frac{s}{a} + \sinh \left( \frac{x_2}{a} \right) \right) \right) - a$ ... combining (2) & (10)

(12) $y_1 + H = \cosh \left( \frac{x_2}{a} \right) - a$ ... combining (3) & (4)

(13) $\cosh \left( \sinh^{-1} \left( \frac{s}{a} + \sinh \left( \frac{x_2}{a} \right) \right) \right) + \frac{H}{a} = \cosh \left( \frac{x_2}{a} \right)$ ... combining (11) & (12)

(14) $\cosh(\sinh^{-1}(x)) = \sqrt{1 + x^2}$ ... hyperbolic identity

(15) $\sqrt{1 + \left( \frac{s}{a} + \sinh \left( \frac{x_2}{a} \right) \right)^2} + \frac{H}{a} = \cosh \left( \frac{x_2}{a} \right)$ ... combining (13) & (14)

(16) $s^2 + 2s \sinh \left( \frac{x_2}{a} \right) = H^2 - 2H \cosh \left( \frac{x_2}{a} \right)$ ... algebraically cleaning up (15)

(17) $x_2 = a \ln \left( \frac{H^2 - s^2 + \sqrt{(s^2-H^2)^2-4a^2\left(H^2-s^2\right)}}{2a(H+s)} \right)$ ... replacing the hyperbolic trig expressions in (16) with their exponential equivalents: \[
\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}
\]

(18) $s = \sinh \left( \ln \left( \frac{H^2 - s^2 + \sqrt{(s^2-H^2)^2-4a^2\left(H^2-s^2\right)}}{2a(H+s)} \right) \right) + \frac{W}{a} - \sinh \left( \ln \left( \frac{H^2 - s^2 + \sqrt{(s^2-H^2)^2-4a^2\left(H^2-s^2\right)}}{2a(H+s)} \right) \right)$ ... combining (9) & (17)

All that remains is the trivial task of solving (18) for $a$ ... which is left as an exercise to the reader ☺
\[ f(x) = \text{acosh} \left( \frac{x}{a} \right) - a \]

Example 1:

- \( W = 15 \text{ ft} \)
- \( H = 6 \text{ ft} \)
- \( S = 20 \text{ ft} \)

\( a \approx 6.098 \)

\( (x_1, y_1) \approx (5.61, 2.77) \)

\( (x_2, y_2) \approx (-9.39, 8.77) \)

Notice the cable can be predicted to sag 2.77 feet from the lower support

Example 2:

- \( W = 15 \text{ ft} \)
- \( H = 6 \text{ ft} \)
- \( S = 18 \text{ ft} \)

\( a \approx 8.609 \)

\( (x_1, y_1) \approx (4.52, 1.21) \)

\( (x_2, y_2) \approx (-10.48, 7.21) \)

Notice the cable can be predicted to sag 1.21 feet from the lower support

As a bonus, let’s apply this to the recommendations made by a zip-line product manufacturer:

https://www.desmos.com/calculator/cg6sq7vrru
Zipline manufacturer recommendations:

1. The slope (H/W) from start to finish should be 3 – 6% ... depending on the type of trolley

2. A cable should sag approximately 2% of its length (S) as measured from the low point ... but they consider the sag (y) as measured when the cable is pulled tight.

The cable will find its lowest sag (largest y) when the two right triangles are similar.

(1) \( \frac{W-x}{H+y} = \frac{x}{y} \) ... similar triangles are proportionate

(2) \( L_1 + L_2 = S \)

(3) \( \sqrt{(W-x)^2 + (H+y)^2} + \sqrt{x^2 + y^2} = S \) ... Pythagorean Theorem

Putting it all together:

Suppose we want a zip-line with a horizontal distance of \( W = 2430 \) ft.

If we choose a trolley that will allow a 4% slope then \(.04 \times 2430 = H \approx 97 \) ft.

And now \(.02 \times S = y\)

Substituting these values for \( W, H \) & \( y \) into equations (1) and (3) and solving the system graphically yields a value for \( S \approx 2437.8 \) ft. (interestingly ... this is only about 6 ft longer than the minimum length)

Thus \(.02 \times 2437.8 = y \approx 48.8 \) ft.

And substituting these values for \( W, H \) and \( S \) back into (18) and solving yields \( a \approx 10096 \) and an unweighted sag \( \approx 32.8 \) ft.